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The toric phases of the $Y^{p,q}$ quivers

Sergio Benvenuti

Scuola Normale Superiore, Pisa and
INFN, Sezione di Pisa, Italy
E-mail: sergio.benvenuti@sns.it

Amihay Hanany and Pavlos Kazakopoulos

Center for Theoretical Physics, Massachusetts Institute of Technology
Cambridge, MA 02139, U.S.A.
E-mail: hanany@mit.edu, noablake@mit.edu

Abstract: We construct all connected toric phases of the recently discovered $Y^{p,q}$ quivers and show their IR equivalence using Seiberg duality. We also compute the R and global U(1) charges for a generic toric phase of $Y^{p,q}$.

Keywords: AdS-CFT and dS-CFT Correspondence, Brane Dynamics in Gauge Theories, Supersymmetry and Duality.

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Contents

1. Introduction

An interesting class of $\mathcal{N}=1$ superconformal gauge theories can be geometrically engineered placing a stack of D3 branes at the apex of a Calabi-Yau cone. These theories are always *quiver gauge theories*, meaning that all the fields transform in a two-index representation of the gauge group. They admit a natural large-N limit, and in this limit the gravitational trace anomalies satisfy the relation $c = a$. The more interesting aspect is that it is possible to take the near horizon limit [1, 2]: there is a string dual, provided by type-IIB string theory on $AdS_5 \times X_5$. X_5 is the compact Einstein base of the six-dimensional cone, which is Calabi-Yau if X_5 is *Sasaki-Einstein*. In order to have a complete description the gauge/string correspondence it is of course desirable to have the explicit knowledge of the background, i.e. of the Sasaki-Einstein metric on X_5.

Until less than a year ago, the explicit metric on X_5 was known only for two homogeneous Sasaki-Einstein manifolds: S^5 and $T^{1,1}$. The first case corresponds to $\mathcal{N}=4$ SYM. The second case, the conifold, was analysed in [3] and corresponds to a $\mathcal{N}=1$ superconformal quiver with gauge group $\mathrm{SU}(N)\times\mathrm{SU}(N)$. Of course it is possible to take orbifolds of these spaces, leading to manifolds with local geometry of S^5 or $T^{1,1}$. A remarkable development in the field of Sasakian-Einstein geometry changed this situation: Gauntlett, Martelli, Sparks and Waldram in [4, 5] found a countably infinite family of explicit non-homogeneous five-dimensional Sasaki-Einstein metrics. The corresponding manifolds are called $Y^{p,q}$, where $q < p$ are positive integers.

Recently, the dual superconformal field theories were constructed [6]. The theories bare the name $Y^{p,q}$ and they are quiver gauge theories. The precise structure of the superpotential was found, allowing a comparison between the global symmetries of the gauge theories and the isometries of the manifolds. An analogous match was performed for the baryonic symmetry. As a further non-trivial check of the gauge/string duality, the volumes of the manifolds and of some supersymmetric three-cycles were computed in field theory and matched with geometric results. This was done using the general field theoretic technique of a-maximization, that was also applied to the known del Pezzo 1 (corresponding to $Y^{2,1}$, [7]) and del Pezzo 2 quivers in [8].

The metric on the $Y^{p,q}$ [4, 5, 7] in local form can be written as:

$$\begin{aligned} \mathrm{d}s_5^2 = & \frac{1-y}{6}(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) + \frac{1}{w(y)q(y)}\mathrm{d}y^2 + \frac{q(y)}{9}(\mathrm{d}\psi - \cos\theta \mathrm{d}\phi)^2 + \\ & + w(y)\left[\mathrm{d}\alpha + f(y)(\mathrm{d}\psi - \cos\theta \mathrm{d}\phi)\right]^2 \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} w(y) &= \frac{2(b-y^2)}{1-y} \\ q(y) &= \frac{b-3y^2+2y^3}{b-y^2} \\ f(y) &= \frac{b-2y+y^2}{6(b-y^2)} \,. \end{aligned} \tag{1.2}$$

The coordinate y ranges between the two smallest roots y_1, y_2 of the cubic $b - 3y^2 + 2y^3$. The parameter b can be expressed in terms of the positive integers p and q:

$$b = \frac{1}{2} - \frac{(p^2-3q^2)}{4p^3}\sqrt{4p^2-3q^2}\,. \tag{1.3}$$

The topology of the five-dimensional $Y^{p,q}$ spaces is $S^2 \times S^3$. The isometry group is $\mathrm{SO}(3) \times \mathrm{U}(1) \times \mathrm{U}(1)$ for both p and q odd, and $\mathrm{U}(2) \times \mathrm{U}(1)$ otherwise. This shows up as global symmetry of the quiver gauge theories. We will not enter into the details of these metrics, and we refer the reader to [7] for an in-depth exposition.

On the other side of the correspondence one finds the $Y^{p,q}$ quiver gauge theories. These were constructed in [6] where it was shown that they can be obtained from the $Y^{p,p}$ theory. The five-dimensional $Y^{p,p}$ space is not smooth, but can be formally added to the list of the $Y^{p,q}$ spaces, and is the base of the $\mathbb{C}^3/\mathbb{Z}_{2p}$ orbifold. The action of the orbifold group on the three coordinates of $\mathbb{C}^3$, $z_i, i = 1, 2, 3$ is given by $z_i \to \omega^{a_i} z_i$ with ω a $2p$-th root of unity, $\omega^{2p} = 1$, and $(a_1, a_2, a_3) = (1, 1, -2)$. The dual gauge theory is easily found. To get the $Y^{p,q}$ theories, one starts from $Y^{p,p}$ and applies an iterative procedure $p - q$ times. We will discuss the details of this method in the next section. At the IR fixed point, one can use Seiberg duality [9] to find an infinite class of theories that are inequivalent in the UV but flow to the same conformal fixed point in the IR. We call these the *phases* of the $Y^{p,q}$ theories. A finite subclass of these are the so-called *toric phases*. These theories have the property that all gauge groups in the quiver have the same rank and every bifundamental field appears in the superpotential exactly twice: once with a positive sign and once with a negative sign. These properties make explicit the fact that the geometry transverse to the D3 branes is toric (hence the name). The IR equivalence of such theories (also called 'toric duality') was discovered in [10], interpreted as Seiberg duality in [11, 12] and further elaborated in [13, 14].

The purpose of this note is to construct all the *connected* toric phases of the $Y^{p,q}$ quivers. These are the toric phases that can be reached by applying Seiberg duality on self-dual gauge groups, i.e. $\mathrm{SU}(N)$ gauge groups with $N_f = 2N_c$ flavors, whose rank remains the same after the duality. Starting from a toric phase, one gets another toric phase by dualising a self-dual node of the quiver. By studying the phases we get from these dualisations we will derive a method for constructing all the connected toric phases of the $Y^{p,q}$ theories as combinations of different types of 'impurities' on the $Y^{p,p}$ quiver. We also demonstrate the agreement between properties of these quivers and geometric predictions by computing the R-charges, and show how one can break conformal invariance (while preserving supersymmetry) by adding fractional branes.

2. The connected toric phases

In this section we construct the connected toric phases of the $Y^{p,q}$ quivers. As mentioned in the introduction, the term 'connected' means the that we are only considering the toric phases that can be reached by applying Seiberg duality on self-dual gauge groups. We do not have a general proof that these are all the toric phases, and it is in principle possible that there are toric phases that can only be reached by going through non-toric ones. However, our experience with a number of examples leads us to believe that this is in fact impossible. For instance, in the case of 3-block and 4-block chiral quivers, the classification of [15] implies that all the toric phases are indeed connected. It will be interesting to find a proof of this. A general property of the toric phases (when they exist, as is the case for the $Y^{p,q}$ quivers), for any superconformal quiver, is that they are always 'minimal models', or 'roots' of the Duality Tree. This can be seen in the following way. By the definition of the toric phase all the ranks of the gauge groups are equal. This implies that the 'relative number of flavors' $\mathrm{n}^{\mathrm{F}} \equiv \frac{N_f}{N_c}$ is always a positive integer number. For instance in the models constructed in [6] one always find $\mathrm{n}^{\mathrm{F}} = 2$ or $\mathrm{n}^{\mathrm{F}} = 3$, meaning that there are gauge groups with $N_f = 2N_c$ or $N_f = 3N_c$. Now, if successive application of Seiberg dualities results in a phase with some $\mathrm{n}^{\mathrm{F}} = 1$, a problem would occur, since the IR of a gauge group with $N_f = N_c$ is not superconformal. This would be a contradiction with the results obtained by Seiberg in [9]. The conclusion is that for any toric phase all the relative number of flavors are integer numbers satisfying $\mathrm{n}^{\mathrm{F}} \geq 2$. This is precisely the condition [15] for a model to be a root of the Duality Tree, i.e. a (local) minimum for the sum of the ranks of the gauge groups. In all the models discussed in this paper, n^{F} will be equal to 2, 3, or 4.

The $Y^{p,q}$ gauge theories can be built starting from $Y^{p,p}$ through an iterative procedure described in detail in [6]. The $Y^{p,p}$ quiver has a particularly simple form. It has $2p$ nodes, each representing an $\mathrm{SU}(N)$ gauge group, that can be placed at the vertices of a polygon. If we number the nodes with an index i, $i = 1, \ldots, 2p$ in a clockwise direction, then between nodes i and $i+1$ there is a double arrow $X_i^\alpha, \alpha = 1, 2$, representing two bifundamental fields that form a doublet of the $\mathrm{SU}(2)$ global symmetry and between nodes i and $i-2$ there is a single arrow Y_i (a singlet of the same $\mathrm{SU}(2)$). For example the quiver for Y^{44} is shown in the upper left corner of figure 1. Following the conventions of [6], we denote the doublets on the outer polygon as $U_i = X_{2i}, V_i = X_{2i+1}$. In figure 1 the U fields are colored

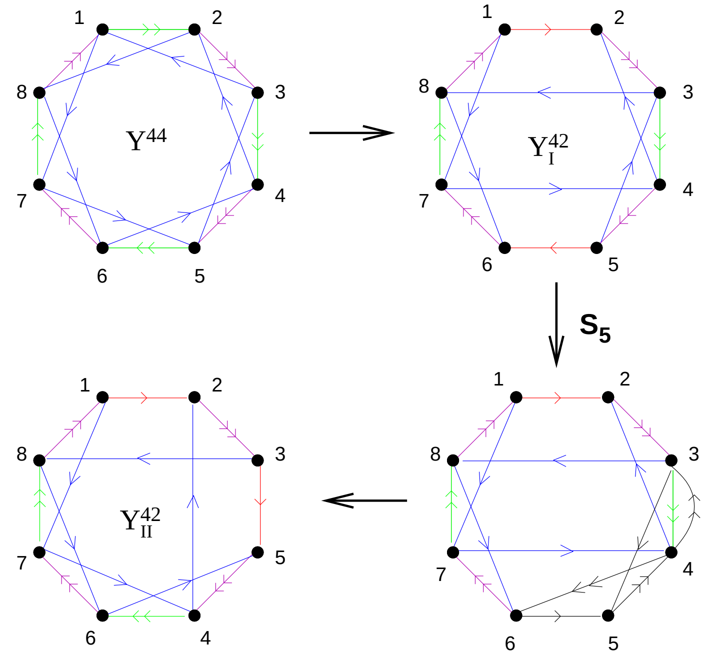

Figure 1: Seiberg duality moves the impurities. The notation S_5 means Seiberg duality on node 5.

cyan, the V fields green and the Y fields blue. The superpotential for this theory consists of all possible cubic terms contracted in a fashion that makes it an invariant of the SU(2) global symmetry. It is written:

$$W = \sum_{i=1}^{p} \epsilon_{\alpha\beta}(U_i^\alpha V_i^\beta Y_{2i+2} + V_i^\alpha U_{i+1}^\beta Y_{2i+3})\,. \tag{2.1}$$

The iterative procedure that produces $Y^{p,q}$ is as follows:

- Pick an edge of the polygon that has a V_i arrow[1] starting at node $2i+1$, and remove one arrow from the corresponding doublet to make it a singlet. Call this type of singlet Z_i.
- Remove the two diagonal singlets, Y that are connected to the two ends of this singlet Z. Since the V_i arrow which is removed starts at node $2i+1$ the Y fields which are removed are Y_{2i+2} and Y_{2i+3}. This action removes from the superpotential the corresponding two cubic terms that involve these Y fields.
- Add a new singlet Y_{2i+3} such that together with the two doublets at both sides of the singlet Z_i, an oriented rectangle is formed. Specifically this arrow starts at node $2i+3$ and ends at node $2i$. The new rectangle thus formed contains two doublets which as before should be contracted to an SU(2) singlet. This term is added to the superpotential.

[1]Picking a V arrow instead of a U is purely a matter of convention, since U and V are equivalent in $Y^{p,p}$.

JHEP07(2005)021

For $Y^{p,q}$ one has to apply the procedure $p-q$ times. For example, a phase of Y^{42} is shown at the upper right side of figure 1. The Z singlets are shown in red. The added Y singlets are shown in blue. That they have the same color and notation is justified by the the fact that, as shown in [6], they have the same R-charge and global U(1) charges as the Y singlets of $Y^{p,p}$. We will use the term 'impurity' for each 3-step substitution in the $Y^{p,p}$ quiver as above. In this language, $Y^{p,q}$ contains $p-q$ impurities. An important point is that $Y^{p,p-1}$, and in general $Y^{p,q}$, is a conformal gauge theory with $c=a$ only at the IR fixed point.

We must emphasize that what we call IR fixed point is really a manifold of fixed points, as also discussed in [16]. On the string theory side, it is possible to modify the background changing the vev of the axion-dilaton, and giving a vev for the complex B-field over the S^2 (there is precisely one such possible vev since the second Betti number of the $Y^{p,q}$ manifolds is always 1). On the gauge theory side this corresponds, respectively, to a simultaneous rescaling of the gauge and superpotential couplings, and to a relative change in the gauge couplings (there is precisely one gauge coupling deformation since the kernel of the quiver matrix is always two). This discussion implies that the conformal manifold is at least two-complex dimensional. It would be nice to see if there are additional marginal directions, corresponding in the gauge theory to exactly marginal superpotential deformations and in the supergravity to continuosly turning on vevs for the other type-IIB forms (these deformations would probably break the SU(2) global symmetry).

Also note that these IR fixed points, for finite $q \neq p$, are not perturbatively accessible. One way to see this is that there are always *finite* anomalous dimensions for the bifundamental fields (and so for all chiral operators), and this is clearly inconsistent with a fixed point were all the couplings are infinitesimal. A simple way to see that there are always finite anomalous dimensions is by noting that in all the phases of the quivers there are always some gauge groups with $\mathrm{n}^{\mathrm{F}}=2$; the numerator of the NSVZ beta function vanishes with infinitesimal anomalous dimensions only if $\mathrm{n}^{\mathrm{F}}=3$. All Seiberg dual phases share the same property, since the chiral spectrum is invariant under Seiberg duality.

2.1 Seiberg duality moves the impurities

The above procedure gives toric phases of $Y^{p,q}$. All nodes (gauge groups) have rank N and every field enters the superpotential exactly twice. However, there is a freedom involved in this construction, namely the choice of positions for the impurities. There are p available positions (the positions of the V doublets) and $p-q$ impurities to distribute. The resulting theories are generally different in the UV. We will now show that they are equivalent at the IR fixed point, related by Seiberg duality. First note that all nodes in $Y^{p,p}$ have $\mathrm{n}^{\mathrm{F}}=3$, so none of them is self-dual. Placing the impurities changes the relative number of flavors from three to two for the nodes at the ends of the Z arrows. So the only self-dual nodes in $Y^{p,q}$ are the ones at the ends of the Z arrows. Dualising any one of these nodes will result in a different toric phase. We illustrate this using the example of Y^{42}. Phase I (the notation is arbitrary) is shown in the upper right side of figure 1. We have four choices on which node to dualise: Nodes 1, 2, 5, 6 are self-dual. We choose node 5 and dualise as usual. The new quarks and mesons are shown in black in the lower right side of figure 1. Note that the mesons M_{43}^{α} and M_{46}^{α} are products of a doublet and a singlet of the SU(2) isometry and thus transform as doublets. The quartic term in the superpotential involving nodes 4, 5, 6, 7 becomes a cubic term with nodes 4, 6, 7, and another cubic term, $\epsilon_{\alpha\beta}X_{54}^{\alpha}M_{46}^{\beta}X_{65}$ is added to the superpotential. The cubic term involving nodes 3, 4, 5 becomes a quadratic term $\epsilon_{\alpha\beta}V_1^{\alpha}M_{43}^{\beta}$ which gives mass to these fields, so it must be integrated out in the IR limit. Integrating out these fields we get a new quartic term involving nodes 2, 3, 5, 4. After eliminating the fields that are integrated out and exchanging nodes 4 and 5 we obtain the quiver shown in the lower left side of figure 1. But this is exactly what we get from a different placement of the two impurities. We can describe the effect of Seiberg duality by saying that the impurity has moved by one step. This was also shown in [16] and put to good use in computing duality cascades for $Y^{p,p-1}$ and $Y^{p,1}$. It is easy to see that if we had dualised node six the impurity would have moved one step in the opposite direction in exactly the same fashion. Also, the result of the dualisation depends only on the fact that there is no impurity between nodes 3 and 4. The rest of the quiver goes along for the ride. So dualising one of the nodes at the ends of a Z field moves the impurity one step in the direction of the dualised node, as long as there is no impurity already there. We have shown that the different phases one gets from applying the iterative procedure are indeed toric duals. This fact was briefly mentioned in [6].

2.2 Double impurities

The next step in constructing the toric phases of $Y^{p,q}$ is to examine what happens when two impurities 'collide'. We saw before how Seiberg duality on a self dual-node moves the impurity by one step. However, when two impurities are adjacent something different happens. We can illustrate this using Y^{42} as an example. It will become clear that the result can be generalized to any $Y^{p,q}$ because the duality affects only the vicinity of the dualised node. We can start from phase II of Y^{42} (figure 2). Nodes 1, 2, 3, 4 are self-dual. Dualising nodes 1 or 4 will move the impurities as before. Dualising nodes 2 or 3 leads to a new phase. We choose to dualise node 3. The new quarks and mesons are shown in black. The quartic terms associated with the impurities become cubic terms with nodes 1, 2, 8 and 2, 4, 5, and new cubic terms $\epsilon_{\alpha\beta}X_{32}^{\alpha}M_{24}^{\beta}X_{43}$ and $\epsilon_{\alpha\beta}X_{32}^{\alpha}M_{28}^{\beta}X_{83}$ are added to the superpotential according to the prescription of Seiberg duality. After rearranging the nodes we see the new phase Y_{III}^{42} in figure 2. The mesons M_{28}^{α} are shown in golden because as we will see they have different R-charges than the fields we have encountered so far. We denote these fields as C^{α}.

This is a new toric phase, different from the ones constructed from the procedure of [6], but equivalent to these at the IR fixed point. An interesting thing to note is that this phase includes only cubic terms in the superpotential (true only for two impurities) and therefore it is a perturbatively renormalizable gauge theory. A closer look at this quiver shows that it actually can be seen formally as a result of applying the procedure of [6] *twice* on the same V doublet. We call this a *double impurity.* So applying Seiberg duality to a self dual node moves the impurity when there is an 'empty slot', but in the case where there is already another impurity there, the two impurities fuse into a double impurity. It is clear that the result of this dualisation does not depend on the rest of the quiver and so it is not specific to Y_{II}^{42}. Two adjacent single impurities can be 'fused' in this fashion in any $Y^{p,q}$.

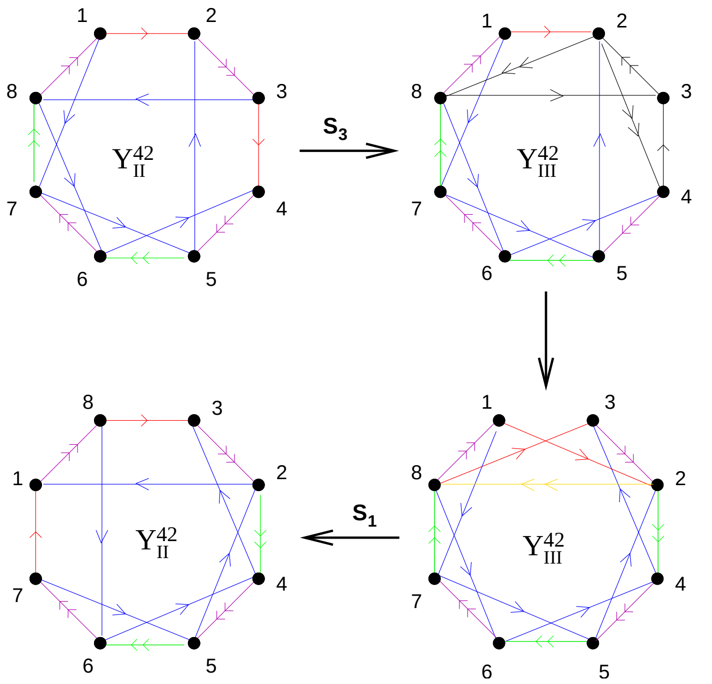

Figure 2: Single impurities can fuse into double impurities.

In Y^{42}_{III}, the only self dual nodes are 1 and 3. Dualising node 3 will lead back to Y^{42}_{II}, since two successive dualisations on the same node always give back the same theory. In exactly the same way, dualising node 1 will break up the double impurity into two adjacent single impurities, giving back the Y^{42}_{II} model.

A picture is starting to emerge: Single impurities can be moved around and fused into double impurities, double impurities can be broken into single impurities and all these models are toric phases. In this fashion one can think of Seiberg duality as the 'motion of free particles on a circle'. It remains to see what happens when double impurities 'collide' with single impurities or other double impurities. The answer is that nothing new happens, and single and double impurities are the only possible configurations in toric phases. Figure 3 illustrates this. We see a phase of Y^{41} with a double impurity next to a single impurity, labeled Y^{41}_{I}. Node 3 has $\mathrm{n}^{\mathrm{F}} = 3$ and dualising it will give a non-toric phase. Nodes 1, 2, 4 are self dual. We already know that dualising node 1 will separate the double impurity into two single ones, and give a model with three single impurities. Dualising 4 will move the single impurity by one step in a clockwise direction. Dualising 2 will also break the double impurity, but the single impurity that is created fuses with the single impurity next to it to give another double impurity. We get a different phase with one single and one double impurity, labeled Y^{41}_{II} in figure 3 (note the rearrangement of nodes 2 and 3). A phase of Y^{40} with two double impurities is also shown in the figure. The only self-dual nodes are 1 and 4. Dualising either one separates the corresponding double impurity as before.

In all these models, all cubic and quartic gauge invariants in the quiver enter the superpotential. Each of these terms contains two SU(2) doublets which are contracted

JHEP07(2005)021

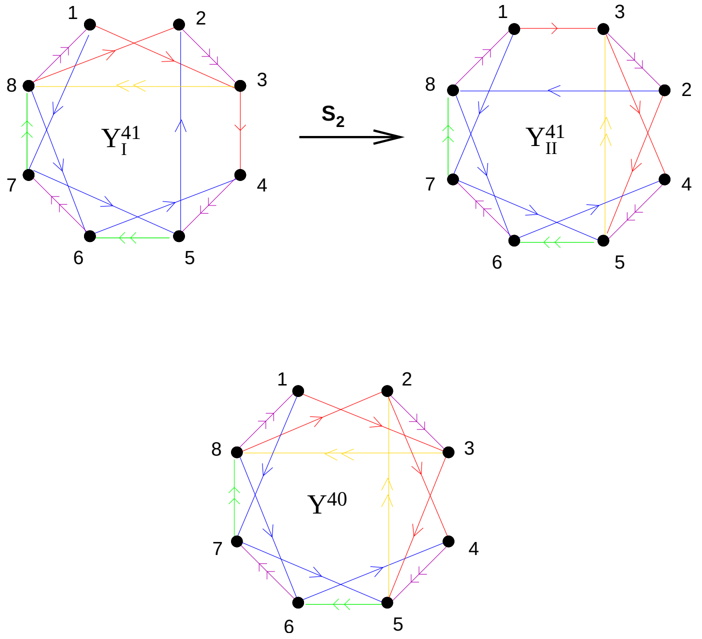

Figure 3: Models with one single and one double impurity and with two double impurities.

into an SU(2) singlet. Single impurities contribute quartic terms, double impurities cubic terms, and we also have the cubic terms carried over from $Y^{p,p}$. We can now state the final result: All connected toric phases of $Y^{p,q}$ can be constructed by placing n_1 single impurities and n_2 double impurities, with $n_1 + 2n_2 = p - q$, on $n_1 + n_2$ of the p available positions[2] of the V doublets of $Y^{p,p}$. We have also seen how Seiberg duality connects all these models by moving, fusing and separating the impurities. It is worth mentioning that those models that contain only double impurities have cubic superpotentials and therefore are renormalizable quantum field theories. We note that a double impurity still 'occupies' 4 nodes of the quiver, in the sense that there are 4 consecutive nodes with $\mathrm{n}^{\mathrm{F}} \neq 3$. This explains why it is impossible to merge together a lot of impurities, and is consistent with the fact that there are only single and double impurities. It is also easy to see that turning on a non-zero vev for the $p-q$ Z fields in all these models higgses the quiver to the one for the orbifold $\mathbb{C}^3/\mathbb{Z}_{p+q}$. We will now proceed to compute the R-charges for a generic toric phase.

3. R-charges for a generic toric phase

We can compute the R-charges of any toric phase using a-maximization [17]. The non-R global symmetry group in all of the models that we have constructed is $\mathrm{SU}(2) \times \mathrm{U}(1)_B \times \mathrm{U}(1)_F$. All fields transform in either singlets or doublets of the SU(2) and are charged under the global U(1)'s. Since there are two U(1)'s with which the R symmetry can mix there will be two unknowns in the a-maximization. Because of the presence of the U(1)-flavor the R-charges can be irrational (if only baryonic U(1) symmetries, with vanishing cubic 't Hooft anomalies, are present, one has to maximize a quadratic function). The trial R-charge

[2]Note that only the relative positions of the impurities matter.

JHEP07(2005)021

Field	number	R-charge	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_F$
Z	$p-q$	$(-4p^2+3q^2-2pq+(2p+q)\sqrt{4p^2-3q^2})/3q^2$	$p+q$	1
Y	$p+q$	$(-4p^2+3q^2+2pq+(2p-q)\sqrt{4p^2-3q^2})/3q^2$	$p-q$	-1
U^α	p	$(2p(2p-\sqrt{4p^2-3q^2}))/3q^2$	$-p$	0
V^α	$p-(n_1+n_2)$	$(3q-2p+\sqrt{4p^2-3q^2})/3q$	q	$+1$
C^α	n_2	$(3q+2p-\sqrt{4p^2-3q^2})/3q$	$-q$	-1

Table 1: Charge assignments for the five different types of fields in the general toric phase of $Y^{p,q}$.

must be anomaly-free (which is equivalent to the vanishing of the NSVZ beta functions for the $2p$ gauge groups) and all terms in the superpotential must have R-charge two.

The following assignment satisfies these conditions:

- The $(p-q)$ singlets Z have R-charge x.
- The $(p+q)$ diagonal singlets Y have R-charge y.
- The p doublets U have R-charge $1-\frac{1}{2}(x+y)$.
- The $p-(n_1+n_2)$ doublets V have R-charge $1+\frac{1}{2}(x-y)$.
- The n_2 doublets C have R-charge $1-\frac{1}{2}(x-y)$

The quiver structure of the gauge theory automatically implies that the linear 't Hooft anomaly $\mathrm{tr}\, R$ vanishes, since it is given by a weighted sum of the gauge coupling beta functions $\mathrm{tr}\, R = \sum N_i \beta_i$ [15]. The cubic 't Hooft anomaly $\mathrm{tr}\, R^3$, proportional to the gravitational central charges $c = a$ [18, 19], is given by:

$$\begin{aligned}
\mathrm{tr}\, R^3_{\mathrm{trial}}(x,y) &= 2p+(p-q)(x-1)^3+(p+q)(y-1)^3-\frac{p}{4}(x+y)^3+ \\
&\quad +\frac{p-n_1-n_2}{4}(x-y)^3-\frac{n_2}{4}(x-y)^3 \\
&= 2p+(p-q)(x-1)^3+(p+q)(y-1)^3-\frac{p}{4}(x+y)^3+\frac{q}{4}(x-y)^3\,.
\end{aligned} \tag{3.1}$$

We have used the relation $n_1+2n_2 = p-q$. The expression for $\mathrm{tr}\, R^3_{\mathrm{trial}}(x,y)$ is the same as the one in [6] and is independent of n_1, n_2. As a consequence, the result is the same for all the toric phases of a given $Y^{p,q}$. This is expected of course, since all toric phases are related by Seiberg duality. The straightforward maximization leads to

$$\begin{aligned}
x_{\mathrm{max}} &= \frac{1}{3q^2}\left[-4p^2-2pq+3q^2+(2p+q)\sqrt{4p^2-3q^2}\right] \\
y_{\mathrm{max}} &= \frac{1}{3q^2}\left[-4p^2+2pq+3q^2+(2p-q)\sqrt{4p^2-3q^2}\right].
\end{aligned} \tag{3.2}$$

The R-charges and global U(1) charges for the fields are shown in table 1. The R-charges and the central charge computed via field theory methods match exactly with the geometric data of the volume of supersymmetric three-cycles and the $Y^{p,q}$ manifolds themselves [6, 7].

The determination of the baryonic charges leads immediately to the determination of the vector of the ranks of the gauge groups in the presence of fractional branes, useful in the study of duality cascades [20 − 22, 16]. The reason is that the $U(1)_B$ symmetry is a linear combination of the $2p$ decoupled gauge U(1)s, corresponding to one of the two null vectors of the quiver matrix. It is important that in table 1 we chose the convention that the baryonic charges are always integers numbers.

The procedure for changing the ranks without developing ABJ anomalies (corresponding to the addition of fractional branes), is very simple and is as follows [23]:

- Start with all the ranks of the gauge groups equal to N. This corresponds to the absence of fractional branes.
- Pick a node I and change the gauge group from $SU(N)$ to, say, $SU(N + M)$.
- Pick an arrow starting from I and arriving at node J. This arrow $I \to J$ has a well defined integer baryonic charge $U(1)_B^{I \to J}$. The rank of the group at node J is precisely $N + M + U(1)_B^{I \to J} M$. For instance, if there is a U-field one has $N + M - pM$, if there is a Z-field one has $N + M + (p + q)M$.
- Pick an arrow starting from J and arriving at node K. Apply the same procedure as above with $U(1)_B^{J \to K}$.
- Go on until all nodes are covered. In case there are only single-impurities, it is enough to do the full loop of lenght $2p$, using the baryonic charges of the doublets U and V and the singlets Z.

It is clear that in this way the new gauge theory, while not conformal if $M \neq 0$, is still free of ABJ gauge anomalies. Of course there are two possible freedoms in the previous construction. First, one can add an "overall" M to the gauge groups, this is equivalent to a shifting in the number of D3 branes at the singularity. Second, it is possible to rescale M, this is equivalent to a rescaling in the number of wrapped D5 branes (or fractional branes).

As check of the procedure, note that after any closed loop one will always find precisely the initial value. This is due to the fact that any "mesonic" operator (corresponding to close loops in the quiver) has vanishing baryonic charge. We note that this simple procedure is valid for any quiver, also in the case where there are more than one U(1)-baryonic symmetries.

4. Conclusions

In this note we have shown how to construct the toric phases of the newly discovered $Y^{p,q}$ quivers using a combination of single and double impurity modifications of $Y^{p,p}$. The impurities move along the circle by each step of Seiberg duality and have the dynamics of free particles on a circle. There is an infinity of Seiberg duals for each of the $Y^{p,q}$ theories, forming a duality tree [20 − 22] and the toric phases lie at the roots of this tree. The natural question in this context is to understand the structure of the full duality tree, including the non toric phases. It would be nice to understand if the various phases are classified by the solutions of some Diophantine equation, as is the case for higher del Pezzo quivers and for all 3 and 4-block chiral models.

Another related problem is the computation of the duality cascades both from the gauge theory and supergravity perspectives. Very significant progress on this has already been made in [16].

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References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] D.R. Morrison and M.R. Plesser, *Non-spherical horizons, I*, *Adv. Theor. Math. Phys.* **3** (1999) 1 [hep-th/9810201].

[3] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B 536** (1998) 199 [hep-th/9807080].

[4] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Supersymmetric AdS_5 solutions of M-theory*, *Class. and Quant. Grav.* **21** (2004) 4335 [hep-th/0402153].

[5] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Sasaki-einstein metrics on $S^2 \times S^3$*, *Adv. Theor. Math. Phys.* **8** (2004) 711 [hep-th/0403002].

[6] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, *An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals*, *JHEP* **06** (2005) 064 [hep-th/0411264].

[7] D. Martelli and J. Sparks, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*, hep-th/0411238.

[8] M. Bertolini, F. Bigazzi and A.L. Cotrone, *New checks and subtleties for AdS/CFT and a-maximization*, *JHEP* **12** (2004) 024 [hep-th/0411249].

[9] N. Seiberg, *Electric - magnetic duality in supersymmetric nonabelian gauge theories*, *Nucl. Phys.* **B 435** (1995) 129 [hep-th/9411149].

[10] B. Feng, A. Hanany and Y.-H. He, *Phase structure of D-brane gauge theories and toric duality*, *JHEP* **08** (2001) 040 [hep-th/0104259].

[11] C.E. Beasley and M.R. Plesser, *Toric duality is Seiberg duality*, *JHEP* **12** (2001) 001 [hep-th/0109053].

[12] B. Feng, A. Hanany, Y.-H. He and A.M. Uranga, *Toric duality as Seiberg duality and brane diamonds*, *JHEP* **12** (2001) 035 [hep-th/0109063].

[13] B. Feng, S. Franco, A. Hanany and Y.-H. He, *Symmetries of toric duality*, *JHEP* **12** (2002) 076 [hep-th/0205144].

[14] B. Feng, A. Hanany, Y.H. He and A. Iqbal, *Quiver theories, soliton spectra and picard-lefschetz transformations*, *JHEP* **02** (2003) 056 [hep-th/0206152].

[15] S. Benvenuti and A. Hanany, *New results on superconformal quivers*, hep-th/0411262.

[16] C.P. Herzog, Q.J. Ejaz and I.R. Klebanov, *Cascading RG flows from new Sasaki-Einstein manifolds*, *JHEP* **02** (2005) 009 [hep-th/0412193].

[17] K. Intriligator and B. Wecht, *The exact superconformal R-symmetry maximizes a*, *Nucl. Phys.* **B 667** (2003) 183 [hep-th/0304128].

[18] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, *Nonperturbative formulas for central functions of supersymmetric gauge theories*, *Nucl. Phys.* **B 526** (1998) 543 [hep-th/9708042].

[19] D. Anselmi, J. Erlich, D.Z. Freedman and A.A. Johansen, *Positivity constraints on anomalies in supersymmetric gauge theories*, *Phys. Rev.* **D 57** (1998) 7570 [hep-th/9711035].

[20] S. Franco, A. Hanany, Y.-H. He and P. Kazakopoulos, *Duality walls, duality trees and fractional branes*, hep-th/0306092.

[21] S. Franco, A. Hanany and Y.-H. He, *A trio of dualities: walls, trees and cascades*, *Fortschr. Phys.* **52** (2004) 540 [hep-th/0312222].

[22] S. Franco, Y.-H. He, C. Herzog and J. Walcher, *Chaotic duality in string theory*, *Phys. Rev.* **D 70** (2004) 046006 [hep-th/0402120].

[23] K. Intriligator and B. Wecht, *Baryon charges in 4D superconformal field theories and their AdS duals*, *Commun. Math. Phys.* **245** (2004) 407 [hep-th/0305046].